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Complete Class Theorems For Unbiased Estimation

EUGENE M. LASKA

PREPARED UNDER
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Introduction

The general problem of unbiased estimation may be stated as follows: For a family of density functions $f(x,\theta)$ find those estimators of θ whose variance is, in some sense, smallest, subject to the restriction that the mean of the estimator be θ.

The sense of minimality that has undergone the most careful scrutiny has been uniform smallness. An unbiased estimator which has uniformly minimum variance is called uniformly efficient (u. e.). Within this framework there are four main questions which have been considered.

1. When do u. e. estimators exist? The problem here does not concern the existence of unbiased estimators. Rather , given that at least one unbiased estimator of finite variance exists (i. e. θ is estimable) when will there be one which is u. e. It is well known [8,9] that in case there exists a real valued sufficient statistic, t, then the conditional expectation of s given t, where s is any unbiased estimator such that the variance of s is finite for each θ, will be u. e. In general, when no such statistic exists, there may or may not be a u. e. estimator and no general theory covers this case.

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2. When are u. e. estimators unique? Uniqueness here refers to essential uniqueness, i. e. two estimators are

the same when the set in the sample space on which they differ has probability zero. In case a real valued, complete, sufficient statistic exists, essential uniqueness is guaranteed. When no such statistic exists, there is no general theory.

3. How does one verify that a given estimator is u. e.? The Cramer-Rao inequality [5] provides a lower bound for the variance of any unbiased estimator. Hence, if any particular estimator has variance uniformly equal to the lower bound, it must be u. e. However, a necessary condition for this to hold is that the estimator be a function of a real valued, complete, sufficient statistic. In this case, we are already guaranteed that it is u. e. In the general situation, it is a difficult matter to show that a given estimator is u. e.

4. Can a purely randomized estimator be u. e.? Within the framework of estimable parameters, this question can be immediately answered in the negative by an appeal to the Rao-Blackwell Theorem [1].

Obviously, for a wide class of distributions, no u. e. estimator exists. Frequently in applications one is concerned primarily with high precision in the neighborhood of a parameter value of particular interest, say θ_0. A particularly appealing criterion is therefore minimization of the variance at θ_0.

Stein [6] successfully studied the problem of locally best unbiased estimation, with reference to the above questions. He gave conditions for existence and uniqueness and gave a functional equation for the determination and verification of such estimators. Randomization is again unnecessary by the Rao-Blackwell Theorem.

One may consider the locally best unbiased estimator as the estimator which has minimum Bayes risk among all unbiased estimators with respect to an a priori measure which puts full weight on the point θ_0. All of the above questions may be asked of Bayes solutions for more general a priori distributions. In the following sections, for compact parameter sets we generalize Stein's results concerning existence and uniqueness (which guarantees admissibility). The final section studies the question of completeness of the class of Bayes solutions.

1. Abstract

Let $f(x,\theta)$ be a given density function continuous in θ, $\theta \in \Omega$, where Ω is a compact subset of the real line. Let $\mathcal{D}$ be the class of randomized unbiased estimators, $\delta(x)$, of θ. For squared error loss function, the class of Bayes solutions is shown to be essentially complete relative to $\mathcal{D}$. If Ω is convex then every purely randomized $\delta(x)$ is inadmissible since by the Rao-Blackwell theorem $E[\delta(x)]$ has smaller risk. A theorem of Stein concerning locally best unbiased estimators is generalized to provide conditions for uniqueness (and therefore admissibility) of Bayes solutions and functional equations for their determination. If the uniqueness condition is satisfied, then the above class is a complete class of admissible unbiased estimators.

2. UNIQUENESS OF BAYES SOLUTIONS

In this section we are concerned with admissible unbiased estimators of a real valued parameter θ of the density function $f(x,\theta)$, $\theta \in \Omega$ where x is a point in the σ-finite measure space $(R,\mathcal{S},\mu)$. We shall assume throughout this paper that Ω is a compact subset of the real line. Let U be the convex set (i. e. $\alpha\delta_1+(1-\alpha)\delta_2 \in U$ if $\delta_1,\delta_2 \in U$, for $0 \leq \alpha \leq 1$) of unbiased estimators of θ. We assume at the outset that U is not empty. We shall be interested in that $\delta^*(x) \in U$ for which

$$r^*(\delta^*,J) = \int_\Omega\int_R(\delta^*(x)-\theta)^2 f(x,\theta)d\mu(x)dJ(\theta)$$

$$= \inf_{\delta \in U} \int_\Omega\int_R (\delta(x)-\theta)^2 f(x,\theta)d\mu(x)dJ(\theta)$$

$$= \inf_{\delta \in U} \int_\Omega \sigma_\theta^2(\delta)dJ(\theta)$$

where $\sigma_\theta^2(\delta) = E_\theta[\delta(x)-\theta]^2$, $J(\theta)$ is any a priori measure over Ω and $\delta^*(x)$ is a Bayes solution with respect to J. The reason for this will be clear after consideration of the following lemma.

LEMMA 1. Suppose $\delta^*(x)$ is the unique estimator minimizing $\int_\Omega \sigma_\theta^2(\delta)dJ(\theta)$ for any distribution function $J(\theta)$ over Ω. Then $\delta^*(x)$ is admissible.

Proof: If $\delta^{*}(x)$ is not admissible there exists an estimator $\delta^{**}(x) \varepsilon U$ with smaller or equal Bayes risk which is a contradiction. Define $\int_{\Omega} f(x,\theta)dJ(\theta) = f^{J}(x)$. We now give a condition for uniqueness of Bayes solutions:

LEMMA 2. If $\int \frac{[f(x,\theta)]^2}{f_J(x)} d\mu(x) < \infty$ for all $\theta \varepsilon \Omega$ and if for some $\delta \varepsilon U$ the Bayes risk $r^{*}(\delta,J)$ is finite, then there exists $\delta^{*}(x)$ such that

$$\int_{\Omega} \sigma_{\theta}^{2}(\delta^{*})dJ(\theta) = \inf_{\delta \varepsilon U} \int_{\Omega} \sigma_{\theta}^{2}(\delta)dJ(\theta).$$

Moreover δ^{*} is essentially unique, i. e., up to an almost everywhere (μ) equivalence.

Proof: For any $\delta \varepsilon U$ the risk is given by

$$r^{*}(\delta,J) = \int_{\Omega}\int_{R}(\delta(x)-\theta)^{2}f(x,\theta)\, d\mu(x)dJ(\theta)$$

$$= \int_{\Omega}\int_{R}\delta^{2}(x)f(x,\theta)\, d\mu(x)dJ(\theta))-\int_{\Omega}\theta^{2}dJ(\theta) < \infty .$$

Since θ belongs to a compact subset of the real line and J is a c. d. f. the last integral is finite and hence the problem is equivalent to minimizing

$$\int_{\Omega}\int_{R}\delta^{2}(x)f(x,\theta)d\mu(x)dJ(\theta) = \int_{R}\delta^{2}(x)f_{J}(x)d\mu(x)$$

subject to

$$\int_R \delta(x) f(x,\theta)\, d\mu(x) \equiv \theta$$

for all $\theta \in \Omega$.

Define the measure ν_J by setting for set $A \in \mathcal{A}$

$$\nu_J(A) = \int_A f_J(x)\, d\mu(x).$$

Our problem is now the equivalent one of minimizing

$$\int_R \delta^2(x)\, d\nu_J(x) \equiv \|\delta\|_J^2,$$

where $\|\phi\|_J$ is the norm on the Hilbert space $L_2(d\nu_J)$ of square integrable functions with respect to the measure ν_J. If δ is unbiased we have,

$$\int_R \delta(x) \frac{f(x,\theta)}{f_J(x)}\, d\nu_J(x) = \left(\delta(x), \frac{f(x,\theta)}{f_J(x)}\right)$$

$$= \theta$$

for all $\theta \in \Omega$ where (p,q) represents the inner product between two functions in $L_2(d\nu_J)$. But now we only wish to find the element in a certain convex subset U of $L_2(d\nu_J)$ which has smallest norm. It is well known that any minimizing sequence $[\delta_n]$ in such a

convex set converges strongly to an essentially unique element $\delta^*(x)$ which has smallest norm [2]. Furthermore strong convergence implies weak convergence. Notice that the first part of the hypothesis of the lemma guarantees that $\frac{f(x,\theta)}{f_J(x)}$ is an element in $L_2(d\nu_J)$. But then since each element in the minimizing sequence has

$$\int_R \frac{f(x,\theta))\delta_n(x)}{f_J(x)}\, d\nu_J(x) = \theta,$$

it follows from weak convergence that $\delta^*(x)$ is unbiased. Thus the proof is complete.

It may be noticed that this proof is similar to that of Stein [6] who considered the case corresponding to a distribution over Ω with all of its mass at one point θ_0, i. e. a locally best unbiased estimator. Stein's other results may also be extended to gain an interesting and useful description of admissible unbiased estimators. We shall illustrate by stating a generalization of his "Principal Theorem".

THEOREM 1. (Generalization of Stein's "Principle Theorem") Suppose that for an a priori distribution $J(\theta)$ over Ω, $\int_R \frac{[f(x,\theta)]^2}{f_J(x)}\, d\mu(x) < \infty$, $\frac{f(x,\theta)}{f_J(x)}$ is finite for all $\theta \in \Omega$ and almost all x and that there exists an unbiased estimator δ of θ for

which $\int \sigma^2_\theta(\delta)dJ(\theta) < \infty$. Then δ^* is the Bayes solution (essentially unique by virtue of Lemma 2) with respect to J if and only if there is a real-valued functional T over the set G of functions of the form

$$\psi(\theta) = \int_R \phi(x)f(x,\theta)d\mu(x)$$

where $\phi(x)$ satisfies

$$\int \phi^2(x)f_J(x)d\mu(x) < \infty$$

(i. e. $\phi \varepsilon L_2(d\nu_J)$), such that

$$T\left(\int \frac{f(x,\theta_1)}{f_J(x)} f(x,\theta)d\mu(x)\right) = \theta_1$$

for all $\theta \varepsilon \Omega$ and

$$T\left(\int \phi(x)f(x,\theta)d\mu(x)\right) = \int \phi(x)\delta^*(x)f_J(x)d\mu(x).$$

Proof: Let $\delta^*(x)$ be the essentially unique unbiased estimator determined in Lemma 2. To show T is defined in a consistent manner, suppose $\phi_1(x)$ and $\phi_2(x)$ are members of G such that

$$\int \phi_1(x)f(x,\theta)d\mu(x) = \int \phi_2(x)f(x,\theta)d\mu(x).$$

Then $\phi = \phi_1 - \phi_2$ satisfies

$$\int \phi(x) f(x,\theta) d\mu(x) = 0 \qquad \text{for all } \theta \in \Omega.$$

We must now show

$$\int \phi(x)\delta^*(x) d\nu_J(x) = 0.$$

Let ε be any real number. Since $\delta^* + \varepsilon\phi$ is an unbiased estimator of θ

$$\begin{aligned} 0 &\leqq \int (\delta^*(x)+\varepsilon\phi)^2 d\nu_J(x) - \int [\delta^*(x)]^2 d\nu_J(x) \\ &= 2\varepsilon \int \phi(x)\delta^*(x) d\nu_J(x) + \varepsilon^2 \int \phi^2(x) d\nu_J(x), \end{aligned}$$

we have, dividing the above by ε and letting $\varepsilon \to 0$,

$\int \phi(x)\delta^*(x) d\nu_J(x) = 0$ as desired.

To prove the converse, suppose $\delta^*(x)$ is any $L_2(d\nu_J)$ function for which there exists such a functional T. Letting

$\phi(x) = \dfrac{f(x,\theta_1)}{f_J(x)}$ gives

$$\int \delta^*(x) f(x,\theta_1) d\mu(x) = T \int \frac{f(x,\theta_1)}{f_J(x)} f(x,\theta) d\mu(x) = \theta_1,$$

and so $\delta^*(x)$ is unbiased. If $\delta(x)$ is any other unbiased estimator of θ with finite risk it is an element of $L_2(d\nu_J)$. Thus

$$T\theta = T\int \delta(x)f(x,\theta)d\mu(x) = \int \delta(x)\delta^*(x)d\nu_J(x)$$

but also

$$T\theta = T\int \delta^*(x)f(x,\theta)d\mu(x) = \int [\delta^*(x)]^2 d\nu_J(x).$$

Applying Schwarz's inequality gives

$$\int [\delta^*(x)]^2 d\nu_J(x) \leqq \int [\delta(x)]^2 d\nu_J(x)$$

with strict equality only if $\delta^*(x)$ is essentially equal to $\delta(x)$.

Q. E. D.

Although we do not do so here, Stein's corollaries, ([6], pp. 409; 410), also can be generalized in a natural manner to provide explicit methods of determination of T and therefore of the Bayes solutions which, at the very least, form an important subset of the admissible class.

Let us suppose θ is a location parameter and consider briefly the class I of all estimators $\delta \in U$ such that the variance $\sigma_\theta^2(\delta)$ is invariant under changes in θ. Then a $\delta^* \in I$ such that $\sigma^2(\delta^*) = \inf_{\delta \in I} \sigma^2(\delta)$, can under certain conditions be shown to be

admissible. We have an immediate corollary to Theorem 1.

COROLLARY 1. Suppose that $f(x,\theta)$ is symmetric in x about θ, where θ is a location parameter, and that Ω is convex. Further suppose the assumptions of Theorem 1 are satisfied for $J(\theta)$, the uniform distribution. Then δ^* is admissible.

Proof: The proof is immediate if we notice that δ^* coincoincides with the well known Pitman estimator [4] which is unbiased by the convexity and symmetry assumptions and which is the unique Bayes solution with respect to the uniform a priori distribution over Ω.

Q. E. D.

3. A Complete Class Theorem

The main content of this section is Theorem 2 which asserts that under certain conditions the class of Bayes solutions is essentially complete. We begin with a few technical lemmas.

LEMMA 3. Let $(\mathcal{R},\mathcal{A},\mu)$ be a σ-finite measure space and $(\delta_n;\ n=0,1,2\ldots)$ be a sequence of $\mathcal{A}$-measurable functions such that

$$\int_B \delta_n d\mu \to \int_B \delta_o d\mu$$

for all $B \in \mathcal{A}$ with $\mu(B) < \infty$. Suppose $|\delta_n(x)| \leq C$ for all $n=0,1,2,\ldots$ $x \in R$, where C is a fixed but arbitrary constant. Then

$$\int_R \delta_n g d\mu \to \int_R \delta_o g d\mu$$

for all $g \in L^1(\mu)$, the space of absolutely integrable functions with respect to the measure μ.

For proof see [2, Problem 6, p. 339].

LEMMA 4. Let Ω be a compact metric space and $(\pi_n; n=0,1,2,\ldots)$ be a sequence of probability measures on Ω. In order that

$$\int_\Omega g(\theta) d\pi_n(\theta) \to \int_\Omega g(\theta) d\pi_o(\theta)$$

for all continuous g on Ω it is sufficient that for some covering net $[D_{k_1\ldots k_m}]$

$$\pi_n(\bar{D}_{k_1\ldots k_m}) \to \pi_o(\bar{D}_{k_1\ldots k_m}) \qquad \text{for all } k_1\ldots k_m.$$

(For the definition of a covering net see [7, p. 66].

Proof: Let g be continuous over Ω. Since

diam $(\bar{D}_{k_1 \ldots k_m}) \to 0$ as $m \to \infty$ uniformly in $k_1 \ldots k_m$, given $\eta > 0$, there exists m_o such that diam $(\bar{D}_{k_1 \ldots k_m}) \leqq \eta$ for all $m \geqq m_o$, all $k_1 \ldots k_m$.

For any fixed $m \geqq m_o$,

$$\int_{\Omega} g d\pi_n = \sum_{k_1, k_2 \ldots k_m} \int_{\bar{D}_{k_1 \ldots k_m}} g d\pi_n .$$

Thus

$$\left| \int_{\Omega} g d\pi_n - \int_{\Omega} g d\pi_o \right| = \left| \sum_{k_1 \ldots k_m} \int_{\bar{D}_{k_1 \ldots k_m}} g d\pi_n - \int_{\bar{D}_{k_1 \ldots k_m}} g d\pi_o \right|$$

Letting

$$\alpha_{k_1 \ldots k_m} = \inf_{\theta' \varepsilon \bar{D}_{k_1 \ldots k_m}} g(\theta'), \quad \beta_{k_1 \ldots k_m} = \sup_{\theta' \varepsilon \bar{D}_{k_1 \ldots k_m}} g(\theta')$$

we have

$$\left| \int_{\Omega} g d\pi_n - \int_{\Omega} g d\pi_o \right| = \left| \sum_{k_1 \ldots k_m} \alpha_{k_1 \ldots k_m} \left(\int_{\bar{D}_{k_1 \ldots k_m}} d\pi_n - \int_{\bar{D}_{k_1 \ldots k_m}} d\pi_o \right) \right|$$

$$+\sum_{k_1\ldots k_m}\left(\int_{\bar{D}_{k_1\ldots k_m}}\left|g-\alpha_{k_1\ldots k_m}\right|d\pi_n+\int_{\bar{D}_{k_1\ldots k_n}}\left|g-\alpha_{k_1\ldots k_n}\right|d\pi_o\right)$$

$$\leq\left|\sum_{k_1\ldots k_m}\alpha_{k_1\ldots k_m}\left(\int_{\bar{D}_{k_1\ldots k_m}}d\pi_n-\int_{\bar{D}_{k_1\ldots k_m}}d\pi_o\right)\right|+2\max_{k_1\ldots k_m}\left(\beta_{k_1\ldots k_m}-\alpha_{k_1\ldots k_m}\right).$$

Since Ω is compact, g is uniformly continuous. Therefore given $\varepsilon > 0$ there exists $\eta_o > 0$ such that

$$\sup_{\theta'\in A} g(\theta')-\inf_{\theta'\in A} g(\theta')<\varepsilon \quad \text{whenever diam } A \leqq \eta_o.$$

Thus, choosing $\eta < \eta_o$

$$\left|\int g\,d\pi_n-\int g\,d\pi_o\right|\leqq\left|\sum_{k_1\ldots k_m}\alpha_{k_1\ldots k_m}\left(\int_{\bar{D}_{k_1\ldots k_m}}d\pi_n-\int_{\bar{D}_{k_1\ldots k_m}}d\pi_o\right)\right|+2\varepsilon$$

whenever $m \geqq m_o$.

Fixing m_o and ε, letting $n \to \infty$, we have

$$\lim_{m \to \infty} \sup \left| \int_{\Omega} g d\pi_n - \int_{\Omega} g d\pi_o \right| \leqq 2\varepsilon$$

for

$$\lim_{m \to \infty} \left| \sum_{k_1 \ldots k_m} a_{k_1 \ldots k_m} \left(\int_{\bar{D}_{k_1 \ldots k_m}} d\pi_n - \int_{\bar{D}_{k_1 \ldots k_m}} d\pi_o \right) \right| = 0, \text{ by hypothesis.}$$

Q. E. D.

LEMMA 5. Let $\delta(A,x)$ be a probability measure in A for each x and measurable for each $A(A \subseteq \Omega,\ x \varepsilon R)$. Let $g(\theta')$ be a bounded measurable function of $\theta' \varepsilon \Omega$. Define (for γ any probability measure on R)

$$\nu(A) = \int_R \delta(A,x) d\gamma(x).$$

Then ν is a probability measure on Ω and

$$\int_{\Omega} g(\theta') d\nu(\theta') = \int_R \left(\int_{\Omega} g(\theta') d\delta(\theta',x) \right) d\gamma(x).$$

Note: Ω being a compact metric space implies that the Borel sets form a σ-algebra.

Proof: See Loeve ([3], Problem 13, page 140).

THEOREM 2. Let Ω be a compact subset of the real line, and $f(x,\theta)$ continuous in θ. Then the class of Bayes solutions with respect to a priori distributions $G(\theta)$ over Ω is essentially complete where the space of decision functions U_r is the class of unbiased, randomized estimators, $\delta(x,A)$. That is, $\delta(x,A)$ is a probability measure over Ω for each x and measurable for each measurable set A, $A \subseteq \Omega$, $x \varepsilon R$. Further, if is convex then any purely randomized decision function is inadmissible.

Proof: It is sufficient to show that the assumptions of Wald ([7], 3.1 to 3.7) are satisfied. Except for the requirement that the class of decision functions be closed in the sense of "regular convergence" ([7], Assumption 3.6 (ii), p.68), the verification of the other requirements is analogous to that in the well-known case of mean squared error loss function without restriction to unbiasedness. Let $\delta(x)$ be a member of the class U_r of randomized, unbiased estimators, i. e., for each x, $\delta(x)$ is a probability measure over Ω. Let $\bar{D}$ be any measurable subset of Ω. For fixed x and a given decision function δ, let us denote the probability that the decision lies in $\bar{D}$ by $\delta(x,\bar{D}) = \int_{\bar{D}} d\delta(x,\theta)$. To say $\delta(x)$ is unbiased is to require that

$$\int_R \left[\int_{\Omega} \theta' d\delta(x,\theta') \right] f(x,\theta)d\mu(x) \equiv \theta \quad \text{for all } \theta \in \Omega .$$

We now show that the class of randomized estimators, U_r is closed in the sense of regular convergence. Let $[\bar{D}_{k_1 k_2 \ldots k_m}]$ be a covering net of Ω and suppose that

$$\lim_{n \to \infty} \int_{R_S} \delta_n(\bar{D}_{k_1 \ldots k_m}, x)d\mu(x) = \int_{R_S} \delta(\bar{D}_{k_1 \ldots k_m}, x)d\mu(x)$$

for every $\bar{D}_{k_1 \ldots k_m}$ in the net and every set $R_S \in \mathcal{A}$ of finite μ measure, where $[\delta_n(x)]$ is a sequence of randomized unbiased estimators. Then we must show that $\delta(x)$ is a member of U_r. Let $\eta = \left\{ \bar{D}_{k_1 \ldots k_m} \right\}$.

From Lemma 3, with c = 1, we get

$$\int_R \delta_n(A,x)g(x)d\mu(x) \to \int_R \delta_o(A,x)g(x)d\mu(x)$$

for all $g \in L_1(\mu)$ and $A \in \eta$.

In particular taking $g(x) = f(x,\theta)$ we have

$$\int_R \delta_n(A,x)f(x,\theta)d\mu(x) \to \int_R \delta_o(a,x)f(x,\theta)d\mu(x)$$

for all θ and all $A \varepsilon \eta$.

Write

$$dP_\theta = f(x,\theta)d\mu$$

and

$$\nu_n^\theta(A) = \int_R \delta_n(A,x)dP_\theta(x).$$

Rewriting the above we have for each arbitrary but fixed θ

$$\nu_n^\theta(A) \to \nu_o^\theta(A) \qquad \text{for all } A \varepsilon \eta .$$

Hence, by lemma 4, for any continuous function $g(\theta')$ over Ω

$$\int_\Omega g(\theta')d\nu_n^\theta(\theta') \to \int_\Omega g(\theta')d\nu_o^\theta(\theta')$$

(where in lemma 4 we have let $\pi_n = \nu_n^\theta$).

Now $\nu_n^\theta(A) = \int_R \delta_n(A,x)dP_\theta(x)$. Letting $\gamma = P_\theta$ in lemma 5 yields

$$\int_\Omega g(\theta')d\nu_n^\theta(\theta') = \int_R \left(\int_\Omega g(\theta')d\delta_n(\theta',x) \right) dP_\theta(x).$$

Therefore

$$\int_R \left(\int_{\Omega} g(\theta')d\delta_n(\theta',x) \right) dP_\theta(x) \to \int_R \left(\int_{\Omega} g(\theta')d\delta_o(\theta',x) \right) dP_\theta.$$

Writing $g(\theta') = \theta'$ for all $\theta' \varepsilon \Omega$ gives

$$\int_R \int_{\Omega} \theta' d\delta(x,\theta')f(x,\theta)d\mu(x) = \theta \quad \text{for all } \theta \varepsilon \Omega$$

and $\delta(x)$ is an unbiased randomized estimator. Thus U_r is closed in the sense of regular convergence and the assumptions of Wald are satisfied. Therefore essential completeness is proved.

Frequently it will not be necessary to use a randomized estimator for one may apply the Rao-Blackwell theorem, ([5], [4]). That is, for any fixed x, $\delta(x)$ defines a probability measure over Ω. Consider for use as an estimator the conditional expected value of θ, under this distribution, given x. If Ω is convex this new non-randomized estimator is unbiased, it has variance less than or equal to that of its generator, the randomized estimator, which is therefore inadmissible. The proof is now complete.

An extension of this theorem to the case pertinent to robustness is immediate. Introduction of the "nuisance" parameter λ disturbs only the property that Wald's assumption 3.7 ([7], p. 96) is satisfied. Let

$$\sum = \left\{ f(x,\theta,\lambda) \right\} \lambda \varepsilon \Lambda, \theta \varepsilon \Omega \text{ and let } f_n \varepsilon \overline{\sum}; \; n = 1,2,3\ldots$$

and

$$t_r(f\ ,f) = \sup_{M \subset R} \left| \int_M [f_n(x,\theta,\lambda) - f(x,\theta,\lambda)]d\mu(x) \right|$$

where M is any μ-measurable subset of R. Assumption 3.7 concerns itself with compactness of $\sum$ in the sense of the metric t_r and the property that if $f_n \to f$ in the sense of t_r, then

$$\sup_{\hat{\theta}\, \varepsilon\, \Omega} |W(f_{n_o},\hat{\theta}) - W(f,\hat{\theta})| \to 0.$$

where W is the loss function $(\theta-\hat{\theta})^2$. Since assumptions 3.1 to 3.6 are not affected by λ we have

COROLLARY 2. If $f(x,\theta,\lambda)$ is a family of density functions, $\lambda\ \varepsilon\ \Lambda$ where Λ contains finitely many points such that for each fixed λ the assumptions of theorem 2 are satisfied, then the class of Bayes solutions with respect to a priori distributions over $\Omega \times \Lambda$ is essentially complete with respect to the class of randomized unbiased estimators.

The proof of the corollary follows immediately from the fact that the assumptions 3.1-3.9 are satisfied in the same way as in the case when Λ contained only one point. We summarize with the following corollary.

COROLLARY 3. If for each J the conditions of lemma 2 and the conditions of theorem 2 are satisfied then the class of Bayes solutions forms an essentially complete class of admissible estimators.

4. Acknowledgment

The author would like to thank Professor Allan Birnbaum and Dr. V. S. Varadarajan for their many helpful comments and criticisms concerning this work which is based on results obtained in a dissertation entitled "A General Theory of Robustness". The theses, which was submitted in partial fulfillment of the requirements for the Ph. D. in mathematics at New York University in 1962, was supported in part by the Office of Naval Research, Contract No. Nonr 285(38).

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